A HOLOGRAPHIC PRINCIPLE FOR THE EXISTENCE OF IMAGINARY KILLING SPINORS

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ABSTRACT. Suppose that $\Sigma = \partial\Omega$ is the n-dimensional boundary, with positive (inward) mean curvature H, of a connected compact $(n+1)$-dimensional Riemannian spin manifold (Ω^{n+1}, g) whose scalar curvature $R \geq -n(n+1)k^2$, for some $k > 0$. If Σ admits an isometric and isospin immersion F into the hyperbolic space $\mathbb{H}^{n+1}_{-k^2}$, we define a quasi-local mass and prove its positivity as well as the associated rigidity statement. The proof is based on a holographic principle for the existence of an imaginary Killing spinor. For $n = 2$, we also show that its limit, for coordinate spheres in an Asymptotically Hyperbolic (AH) manifold, is the mass of the (AH) manifold.

Dédié à Paul Gauduchon *en témoignage de notre reconnaissance et amitié.*

1. INTRODUCTION

The Positive Mass Theorem (PMT) states that for a complete asymptotically flat manifold which, near each end, behaves like the Euclidean space at infinity and whose scalar curvature is nonnegative, then its ADM mass of each end is non-negative. Moreover, if the ADM mass of one end is zero, then the manifold is the Euclidean space. The PMT was proved by Schoen and Yau [SY1, SY2] using minimal surface techniques. Later on, Witten [Wi] gave an elegant and simple proof of the PMT for spin manifolds. Since then, spinors has been successfully used to prove Positive Mass type theorems (see for example [AD, He, CH, LY1, LY2, Wa1, ST1, HM1]).

In this spirit, Wang and Yau [WY1] introduced a quasi-local mass for 3-dimensional manifolds with boundary whose scalar curvature is bounded from below by a negative constant. Again, using spinorial methods, they proved that this mass is non-negative. Shi and Tam [ST2] proved a similar result but with a simpler and more explicit definition of the mass. More precisely:

Theorem 1. [ST2] *Let (Ω^3, g) be a compact 3-dimensional orientable Riemannian manifold with smooth boundary Σ. Assume that :*

(1) *The scalar curvature R of (Ω, g) satisfies $R \geq -6k^2$ for some $k > 0$;*
(2) *The boundary Σ is a topological sphere with Gauss curvature $K > -k^2$ and mean curvature $H > 0$ (so that Σ can be isometrically embedded into $\mathbb{H}^3_{-k^2}$, the Hyperbolic space of constant curvature $-k^2$, with mean curvature H_0).*

Then, the energy-momentum vector

$$\mathcal{M}_\alpha := \int_\Sigma (H_0 - H) \mathbf{W}_\alpha \, d\Sigma \in \mathbb{R}^{3,1}$$

is future directed non-spacelike or zero, where $\mathbf{W}_\alpha = (x_1, x_2, x_3, \alpha t)$ with

$$\alpha = \coth R_1 + \frac{1}{\sinh R_1} \Big(\frac{\sinh^2 R_2}{\sinh^2 R_1} - 1 \Big)^{\frac{1}{2}} > 1 \tag{1}$$

an explicit constant depending on the intrinsic geometry of Σ and $\mathbf{X} := \mathbf{W}_1 = (x_1, x_2, x_3, t)$ is the position vector in $\mathbb{R}^{3,1}$. Moreover, if there exists a future directed null vector $\zeta \in \mathbb{R}^{3,1}$ such that:

$$\langle \mathcal{M}_\alpha, \zeta \rangle_{\mathbb{R}^{3,1}} = 0,$$

then (Ω^3, g) is a domain in $\mathbb{H}^3_{-k^2}$.

The statement of this result needs some explanation. First, from [P] and [DCW], as mentioned, the assumptions on the boundary Σ ensure the existence of an isometric embedding of Σ into the hyperbolic space $\mathbb{H}^3_{-k^2}$ as a convex surface which bounds a domain D in $\mathbb{H}^3_{-k^2}$. Moreover, this embedding is unique up to an isometry of $\mathbb{H}^3_{-k^2}$. Here H_0 denotes the mean curvature of this embedding and R_1 and R_2 are two positive real numbers such that $B_o(R_1) \subset D \subset B_o(R_2)$ in $\mathbb{H}^3_{-k^2}$ where $B_o(r)$ is the geodesic ball of radius $r > 0$ and center $o = (0, 0, 0, 1/k)$. This result has been recently generalized by Kwong [K]. Namely, he proves:

Theorem 2. [K] *For $n \geq 2$, let (Ω^{n+1}, g) be a compact spin $(n + 1)$-dimensional manifold with smooth boundary Σ. Assume that:*

(1) *The scalar curvature R of Ω satisfies $R \geq -n(n + 1)k^2$ for some $k > 0$,*
(2) *The boundary Σ is topologically an n-sphere with sectional curvature $K > -k^2$, mean curvature $H > 0$ and that Σ can be isometrically embedded uniquely into $\mathbb{H}^{n+1}_{-k^2}$ with mean curvature H_0.*

Then, there is a future time-like vector-valued function $\mathbf{W}_\alpha$ on Σ such that the energy-momentum vector:

$$\mathcal{M}_\alpha := \int_\Sigma (H_0 - H) \mathbf{W}_\alpha \, d\Sigma \in \mathbb{R}^{n+1,1}$$

is future non-spacelike. Here $\mathbf{W}_\alpha = (x_1, x_2, \cdots, x_{n+1}, \alpha t)$ *for some* $\alpha > 1$ *and* $\mathbf{X} := \mathbf{W}_1(x_1, x_2, \cdots, x_{n+1}, t) \in \mathbb{H}^{n+1}_{-k^2} \subset \mathbb{R}^{n+1,1}$ *is the position vector of the embedding of* Σ.

In the general case, the constant α is still explicitly given by (1). It is conjectured, and verified for $n = 2$ in certain cases (see [ST2]), that Theorems 1 and 2 should hold for $\alpha = 1$. A key ingredient in the proof of these two results is a generalization of the Positive Mass Theorem for (AH) manifolds (see Section 4.3).

In this paper, we make use of another approach, developed in [HM1], to establish a holographic principle[1] for the existence of imaginary Killing spinors on Dirac bundles (See Section 2.2) in order to generalize the above results in several directions. Namely, we modify the curvature term in the definition of $\mathcal{M}_\alpha$ to precisely define a energy-momentum vector field $\mathbf{E}(\Sigma)$ in terms of $\mathbf{X}$. In particular, our expression depends only on the metric and the embedding of Σ and is thus independent of the particular manifold Ω. It could be considered as a possible new definition of a quasi-local mass since it has the desirable *non negativity* and *rigidity* properties as shown in Theorems 3 and 4. Moreover, these statements hold in a more general setup.

In fact, we have:

Theorem 3. *Let* (Ω^{n+1}, g) *be a compact, connected* $(n + 1)$-*dimensional Riemannian spin manifold with smooth boundary* Σ. *Assume that*

(1) *The scalar curvature* R *of* Ω *satisfies* $R \geq -n(n + 1)k^2$ *for some* $k > 0$;
(2) *The boundary* $\Sigma = \partial\Omega$ *has mean curvature* $H > 0$ *and that there exists an isometric and isospin immersion* F *of* Σ *into the hyperbolic space* $\mathbb{H}^{n+1}_{-k^2}$ *with mean curvature* H_0.

Then, the energy-momentum vector defined by

$$\mathbf{E}(\Sigma) := \int_\Sigma \Big(\frac{H_0^2 - H^2}{H}\Big)\mathbf{X}\, d\Sigma \in \mathbb{R}^{n+1,1} \tag{2}$$

is timelike future directed or zero (see Theorem 2 for the definition of X). *Moreover,* $\mathbf{E}(\Sigma) = 0$ *if and only if* (Ω^{n+1}, g) *is a domain in* $\mathbb{H}^{n+1}_{-k^2}$, Σ *is connected and the embedding of* Σ *in* Ω *and its immersion* F *in* $\mathbb{H}^{n+1}_{-k^2}$ *are congruent.*

For $n = 2$, since Ω is automatically spin, we deduce the following:

Theorem 4. *Let* (Ω^3, g) *be a compact, connected* 3-*dimensional oriented Riemannian manifold with smooth boundary* Σ. *Assume that:*

(1) *The scalar curvature* R *of* (Ω^3, g) *satisfies* $R \geq -6k^2$ *for some* $k > 0$;
(2) *The boundary* Σ *is a topological sphere with Gauss curvature* $K > -k^2$ *and with mean curvature* $H > 0$.

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[1]By holographic principle we mean the property which states that the description of a manifold with boundary can be thought of as encoded on the boundary.

Then, the energy-momentum vector given by $\mathbf{E}(\Sigma) \in \mathbb{R}^{3,1}$ *is timelike future directed or zero. Moreover,* $\mathbf{E}(\Sigma) = 0$ *if and only if* (Ω^3, g) *is a domain in* $\mathbb{H}^3_{-k^2}$ *and* Σ *is connected.*

Note that in the rigidity part of this result, the embedding of Σ in Ω and its immersion in $\mathbb{H}^3_{-k^2}$ are automatically congruent because of the uniqueness of the embedding of Σ in $\mathbb{H}^3_{-k^2}$.

For simplicity, we will only prove the case $k = 1$. The general case is obtained by a homothetic change of the metric.

2. Geometric and Analytic preliminaries

The aim of this section is to introduce the general geometrical spinorial setting and the basic analytical tools needed to establish the above mentioned results.

2.1. **The geometric setting.** In the following, we consider a compact and connected Riemannian spin $(n + 1)$-dimensional manifold (Ω^{n+1}, g) with smooth boundary $\Sigma := \partial\Omega$. The Riemannian structure on Ω induces a Riemannian metric on Σ, also denoted by g, whose Levi-Civita connection ∇^Σ satisfies the Riemannian Gauss formula

$$\nabla^\Sigma_X Y = \nabla^\Omega_X Y - g\big(A(X), Y\big)N \tag{3}$$

for all $X, Y \in \Gamma(T\Sigma)$. Here ∇^Ω is the Levi-Civita connection on Ω, N the unit inner normal vector field to Σ and A is the Weingarten map defined by $A(X) = -\nabla^\Omega_X N$, for $X \in \Gamma(T\Sigma)$. Since Ω is spin, there exists a pair $(\mathrm{Spin}(\Omega), \eta)$ where $\mathrm{Spin}(\Omega)$ is a Spin_{n+1}-principal fiber bundle over Ω and η is a 2-fold covering of the SO_{n+1}-principal bundle $\mathrm{SO}(\Omega)$ of g-orthonormal frames such that

$$\forall u \in \mathrm{Spin}(\Omega), \ \forall a \in \mathrm{Spin}_{n+1}, \qquad \eta(ua) = \eta(u)\rho(a)$$

where ua denotes the right action of Spin_{n+1} on $\mathrm{Spin}(\Omega)$ and ρ is the 2-fold covering of the special orthogonal group SO_{n+1} by Spin_{n+1}. Note that since Ω is oriented, it induces an orientation on the boundary, hence Σ is automatically spin. Indeed, via the inclusion map $\mathrm{SO}(\Sigma) \hookrightarrow \mathrm{SO}(\Omega)$, we can define the pulled-back bundle $\mathrm{Spin}(\Omega)_{|\Sigma}$, which gives a spin structure on Σ denoted by $\mathrm{Spin}(\Sigma)$. Recall that on Ω, we define the spinor bundle $\mathbb{S}\Omega$, a rank $2^{\left[\frac{n+1}{2}\right]}$ complex vector bundle, by

$$\mathbb{S}\Omega := \mathrm{Spin}(\Omega) \times_{\gamma_{n+1}} \mathcal{S}_{n+1}$$

where γ_{n+1} is the restriction to Spin_{n+1} of an irreducible complex representation of the complex Clifford algebra $\mathbb{C}l_{n+1}$. This representation provides a left Clifford module

$$\gamma^\Omega : \mathbb{C}l(\Omega) \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{S}\Omega) \tag{4}$$

which is a fiber preserving algebra morphism. Then $\mathbb{S}\Omega$ becomes a bundle of complex left modules over the Clifford bundle $\mathbb{C}l(\Omega)$. In particular, $\mathbb{S}\Omega$ is a complex *Dirac bundle* in the sense of [LM], i.e., there exists on $\mathbb{S}\Omega$:

    - a Hermitian scalar product $\langle\,,\,\rangle_\Omega$,
    - a spin Levi-Civita connection ∇^Ω acting on sections of $\mathbb{S}\Omega$

such that

    - the Clifford multiplication by tangent vector fields is skew-Hermitian:

$$\langle\gamma^\Omega(X)\psi,\varphi\rangle = -\langle\psi,\gamma^\Omega(X)\varphi\rangle, \tag{5}$$

    - the covariant derivative ∇^Ω is a module derivation, that is

$$\nabla^\Omega_X\big(\gamma^\Omega(Y)\psi\big) = \gamma^\Omega(\nabla^\Omega_X Y)\psi + \gamma^\Omega(Y)\nabla^\Omega_X\psi, \tag{6}$$

    - the covariant derivative ∇^Ω is compatible with the Hermitian scalar product, that is

$$X\langle\psi,\varphi\rangle = \langle\nabla^\Omega_X\psi,\varphi\rangle + \langle\psi,\nabla^\Omega_X\varphi\rangle \tag{7}$$

for all $X,Y \in \Gamma(T\Omega)$ and $\psi,\varphi \in \Gamma(\mathbb{S}\Omega)$. The Dirac operator D^Ω on $\mathbb{S}\Omega$ is the first order elliptic differential operator locally given by

$$D^\Omega = \sum_{i=1}^{n+1}\gamma^\Omega(e_i)\nabla^\Omega_{e_i},$$

where $\{e_1,\dots,e_{n+1}\}$ is a local orthonormal frame of $T\Omega$. As mentioned above, the boundary is naturally endowed with a spin structure and the group $\mathrm{Spin}_n \subset \mathbb{C}l_n^0$ (the even part of the Clifford algebra) acts on the restricted bundle $\mathrm{Spin}(\Omega)_{|\Sigma}$ via the map ι defined by

$$\begin{aligned}\iota\colon \mathbb{C}l_n &\longrightarrow \mathbb{C}l_{n+1}^0 \subset \mathbb{C}l_{n+1} \\ e_j &\longmapsto e_j\cdot N.\end{aligned}$$

where the dot is the multiplicative structure of the Clifford algebra. Hence we have that the restriction

$$\mathbb{S}\!\!\!/\,\Sigma := \mathbb{S}\Omega_{|\Sigma} = \mathrm{Spin}(\Sigma)\times_{\gamma_{n+1}\circ\iota}\mathcal{S}_{n+1}$$

is a left module over $\mathbb{C}l(\Sigma)$ with Clifford multiplication

$$\gamma\!\!\!/\,^\Sigma : \mathbb{C}l(\Sigma)\longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{S}\!\!\!/\,\Sigma)$$

given by $\gamma\!\!\!/\,^\Sigma = \gamma^\Omega\circ\iota$, that is

$$\gamma\!\!\!/\,^\Sigma(X)\psi = \gamma^\Omega(X)\gamma^\Omega(N)\psi \tag{8}$$

for every $\psi \in \Gamma(\mathbb{S}\!\!\!/\,\Sigma)$ and $X \in \Gamma(T\Sigma)$. Consider on $\mathbb{S}\!\!\!/\,\Sigma$ the Hermitian metric $\langle\,,\,\rangle_\Omega$ induced from that of $\mathbb{S}\Omega$. This metric immediately satisfies the compatibility condition (5) if one puts on Σ the Riemannian metric induced from Ω and the extrinsic Clifford multiplication $\gamma\!\!\!/\,^\Sigma$ defined in (8). Now the

Gauss formula (3) implies that the spin connection $\overline{\nabla}^\Sigma$ on $\slashed{S}\Sigma$ is given by the following spinorial Gauss formula

$$\overline{\nabla}^\Sigma_X \psi = \nabla^\Omega_X \psi - \frac{1}{2}\slashed{\gamma}^\Sigma(AX)\psi \qquad (9)$$

for every $\psi \in \Gamma(\slashed{S}\Sigma)$ and $X \in \Gamma(T\Sigma)$. The extrinsic Dirac operator $\slashed{D}^\Sigma := \slashed{\gamma}^\Sigma \circ \overline{\nabla}^\Sigma$ on Σ defines a first order elliptic operator acting on sections of $\slashed{S}\Sigma$. By (9), for any spinor field $\psi \in \Gamma(\mathbb{S}\Omega)$, we have

$$\slashed{D}^\Sigma \psi = \sum_{j=1}^n \slashed{\gamma}^\Sigma(e_j)\overline{\nabla}^\Sigma_{e_j}\psi = \frac{n}{2}H\psi - \gamma^\Omega(N)D^\Omega\psi - \nabla^\Omega_N\psi, \qquad (10)$$

and

$$\slashed{D}^\Sigma\big(\gamma^\Omega(N)\psi\big) = -\gamma^\Omega(N)\slashed{D}^\Sigma\psi \qquad (11)$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of $T\Sigma$ and $H = \frac{1}{n}\text{trace } A$ is the mean curvature of Σ in Ω. On the other hand, Σ has also an *intrinsic* spinor bundle defined from its spin structure and an irreducible representation of $\mathbb{C}l_n$. More precisely, the complex vector bundle of rank $2^{[\frac{n}{2}]}$, defined by

$$\mathbb{S}\Sigma := \text{Spin}(\Sigma) \times_{\gamma_n} \mathcal{S}_n.$$

This is also a Dirac bundle over Σ with a Clifford multiplication γ^Σ, a spin Levi-Civita connection ∇^Σ and a Hermitian scalar product satisfying the properties (5), (6) and (7) on Σ. Moreover, the intrinsic Dirac operator on Σ is then defined by $D^\Sigma := \gamma^\Sigma \circ \nabla^\Sigma$. As we shall see in the next section, there are natural identifications between intrinsic and extrinsic spinor bundles over Σ (see [Bu, Tr, Bä, HMZ2, HMR2] for more details).

2.2. Dirac bundles and chirality operator.
The important fact now is to consider bundles on which a chirality operator is defined. Recall that a chirality operator ω on a Dirac bundle $(\mathcal{E}\Omega, \gamma, \nabla, \langle\,,\,\rangle)$ is an endomorphism

$$\omega : \Gamma(\mathcal{E}\Omega) \longrightarrow \Gamma(\mathcal{E}\Omega)$$

such that

$$\omega^2 = \text{Id}_{\mathcal{E}\Omega}, \qquad \langle\omega\Psi, \omega\Phi\rangle = \langle\Psi, \Phi\rangle, \qquad (12)$$

$$\omega(\gamma(X)\Psi) = -\gamma(X)\omega\Psi, \qquad \nabla_X(\omega\Psi) = \omega(\nabla_X\Psi), \qquad (13)$$

for all $X \in \Gamma(T\Omega)$ and $\Psi, \Phi \in \Gamma(\mathcal{E}\Omega)$. In the following, we consider the vector bundle given by

$$\mathcal{E}\Omega := \begin{cases} \mathbb{S}\Omega & \text{if } n+1 = 2m, \\ \mathbb{S}\Omega \oplus \mathbb{S}\Omega & \text{if } n+1 = 2m+1, \end{cases}$$

on which a Clifford multiplication γ and a linear connection ∇ are defined by

$$\gamma = \begin{cases} \gamma^\Omega & \text{if } n+1 = 2m \\[2mm] \gamma^\Omega \oplus -\gamma^\Omega = \begin{pmatrix} \gamma^\Omega & 0 \\ 0 & -\gamma^\Omega \end{pmatrix} & \text{if } n+1 = 2m+1 \end{cases} \tag{14}$$

and

$$\nabla = \begin{cases} \nabla^\Omega & \text{if } n+1 = 2m \\[2mm] \nabla^\Omega \oplus \nabla^\Omega = \begin{pmatrix} \nabla^\Omega & 0 \\ 0 & \nabla^\Omega \end{pmatrix} & \text{if } n+1 = 2m+1. \end{cases} \tag{15}$$

Finally, $\langle\,,\,\rangle$ denotes the Hermitian scalar product given by $\langle\,,\,\rangle_\Omega$ for n odd and by

$$\langle \Psi\,,\Phi \rangle := \langle \psi_1, \varphi_1 \rangle_\Omega + \langle \psi_2, \varphi_2 \rangle_\Omega \tag{16}$$

for n even, for any $\Psi = (\psi_1, \psi_2)$, $\Phi = (\varphi_1, \varphi_2) \in \Gamma(\mathcal{E}\Omega)$. The Dirac-type operator acting on sections of $\mathcal{E}\Omega$ and defined by $D := \gamma \circ \nabla$ is explicitly given by

$$D = \begin{cases} D^\Omega & \text{if } n+1 = 2m \\[2mm] D^\Omega \oplus -D^\Omega = \begin{pmatrix} D^\Omega & 0 \\ 0 & -D^\Omega \end{pmatrix} & \text{if } n+1 = 2m+1. \end{cases}$$

Let us examine, in more details, this bundle and its restriction to Σ:

The even dimensional case

If $n+1 = 2m$, the vector bundle $\mathcal{E}\Omega$ is the spinor bundle $\mathbb{S}\Omega$. In this situation, it is well-known that the Clifford multiplication $\omega := \gamma(\omega_{n+1}^{\mathbb{C}})$ by the complex volume element

$$\omega_{n+1}^{\mathbb{C}} := i^m e_1 \cdot \ldots \cdot e_{n+1}$$

defines a chirality operator on $\mathcal{E}\Omega$. Moreover, the spinor bundle splits into

$$\mathcal{E}\Omega = \mathbb{S}\Omega = \mathbb{S}^+\Omega \oplus \mathbb{S}^-\Omega \tag{17}$$

where $\mathbb{S}^\pm\Omega$ are the ± 1-eigenspace of the endomorphism ω. On the other hand, from algebraic considerations (see [HMZ1] or [HMR2] for example) the restricted spinor bundle

$$\mathcal{E} := \mathcal{E}\Omega_{|\Sigma} = \mathbb{S}\Omega_{|\Sigma} = \not{\mathbb{S}}\Sigma$$

can be identified with the intrinsic data of Σ as follows:

$$(\not{\mathbb{S}}\Sigma, \not{\gamma}^\Sigma, \not{\nabla}^\Sigma) \cong (\mathbb{S}\Sigma \oplus \mathbb{S}\Sigma, \gamma^\Sigma \oplus -\gamma^\Sigma, \nabla^\Sigma \oplus \nabla^\Sigma).$$

In the following, for simplicity we let $(\mathcal{E}, \not{\gamma}, \not{\nabla}) := (\not{\mathbb{S}}\Sigma, \not{\gamma}^\Sigma, \not{\nabla}^\Sigma)$ the extrinsic Dirac bundle over the boundary of the even dimensional Riemannian spin domain Ω. As a consequence of these identifications, we get that the extrinsic Dirac-type operator $\not{D} := \not{\gamma} \circ \not{\nabla}$ of $\mathcal{E}$ can be identified with the extrinsic Dirac

operator $\not{D}^\Sigma$ which only depends on the Riemannian and the spin structure of Σ since we also have the following identification

$$\not{D} = D^\Sigma \oplus -D^\Sigma = \begin{pmatrix} D^\Sigma & 0 \\ 0 & -D^\Sigma \end{pmatrix}.$$

Moreover, a simple but important observation here is that we can also choose the Clifford action of the unit normal N by:

$$\gamma(N) = -i \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \tag{18}$$

where the matrix blocks are defined with respect to the chiral decomposition (17). Then we note that the Dirac-type operator defined for all $\Psi \in \Gamma(\mathscr{E})$ by

$$\not{D}^\pm \Psi := \not{D}\Psi \pm \frac{n}{2} i \gamma(N)\Psi = \not{D}^\Sigma \pm \frac{n}{2} i \gamma^\Omega(N)\Psi$$

does not depend on the extrinsic geometry of Σ in Ω. Indeed, from the identification of $\not{D}$ and (18), we have

$$\not{D}^\pm = \begin{pmatrix} D^\Sigma & \pm\frac{n}{2}\mathrm{Id} \\ \pm\frac{n}{2}\mathrm{Id} & -D^\Sigma \end{pmatrix}$$

and it is obvious from this expression that these operators only depend on intrinsic data of Σ (more precisely on the spin structure and the induced metric of Σ).

The odd dimensional case

If $n + 1 = 2m + 1$, the vector bundle $\mathcal{E}\Omega$ consists of two copies of the spinor bundle

$$\mathcal{E}\Omega = \mathbb{S}\Omega \oplus \mathbb{S}\Omega$$

and its rank on $\mathbb{C}$ is 2^{m+1}. It is straightforward from the definitions (14), (15) and (16) that the relations (5), (6) and (7) are valid for γ, ∇ and $\langle\,,\,\rangle$ and thus $(\mathcal{E}\Omega, \gamma, \nabla, \langle\,,\,\rangle)$ defines a Dirac bundle over Ω. In this situation, it is a simple exercise to check that the map

$$\begin{array}{rcl} \omega: \quad \Gamma(\mathcal{E}\Omega) & \longrightarrow & \Gamma(\mathcal{E}\Omega) \\ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} & \longmapsto & \omega\Psi := \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}, \end{array}$$

satisfies the properties (12) and (13) so that it defines a chirality operator on $\mathcal{E}\Omega$. The restriction of $\mathcal{E}\Omega$ to Σ is given by

$$\mathscr{E} := \mathcal{E}\Omega_{|\Sigma} = \not{\mathbb{S}}\Sigma \oplus \not{\mathbb{S}}\Sigma$$

and can be identified with two copies of the intrinsic spinor bundle of Σ (see [HMZ1] or [HMR2] for more details). Similarly, the extrinsic spin Levi-Civita connection

$$\not{\nabla} := \not{\nabla}^\Sigma \oplus \not{\nabla}^\Sigma = \begin{pmatrix} \not{\nabla}^\Sigma & 0 \\ 0 & \not{\nabla}^\Sigma \end{pmatrix} \tag{19}$$

as well as its Clifford multiplication

$$\gamma\!\!\!/ := \gamma\!\!\!/^\Sigma \oplus \gamma\!\!\!/^\Sigma = \begin{pmatrix} \gamma\!\!\!/^\Sigma & 0 \\ 0 & \gamma\!\!\!/^\Sigma \end{pmatrix} \tag{20}$$

are such that the following identifications hold

$$(\mathcal{E}, \gamma\!\!\!/, \overline{\nabla}\!\!\!\!/) \cong (\$\Sigma \oplus \$\Sigma, \gamma^\Sigma \oplus \gamma^\Sigma, \nabla^\Sigma \oplus \nabla^\Sigma).$$

In particular, these definitions provide a Dirac bundle structure on $\mathcal{E}$. It is also clear from the definitions of ∇, $\overline{\nabla}\!\!\!\!/$ and the spinorial Gauss formula (9) that a similar relation holds between ∇ and $\overline{\nabla}\!\!\!\!/$. The extrinsic Dirac-type operator acting on sections of $\mathcal{E}$ is defined as usually by $D\!\!\!\!/ := \gamma\!\!\!/ \circ \overline{\nabla}\!\!\!\!/$ and by (19) and (20), it satisfies:

$$D\!\!\!\!/ = \begin{pmatrix} D^\Sigma & 0 \\ 0 & D^\Sigma \end{pmatrix}.$$

Then we also easily observe that relations (10) and (11) hold. Finally, as in the even dimensional case, the operators defined by

$$D\!\!\!\!/^\pm := D\!\!\!\!/ \pm \frac{n}{2} i\gamma(N) \tag{21}$$

can be expressed intrinsically with respect to Σ. Indeed, by (14), we first note that

$$D\!\!\!\!/^\pm = \begin{pmatrix} D^\Sigma \pm \frac{n}{2} i\gamma^\Omega(N) & 0 \\ 0 & D^\Sigma \mp \frac{n}{2} i\gamma^\Omega(N) \end{pmatrix}.$$

Moreover, since

$$D^\Sigma : \Gamma\big(\mathbb{S}^\pm(\Sigma)\big) \longrightarrow \Gamma\big(\mathbb{S}^\mp(\Sigma)\big)$$

and since we can choose the Clifford multiplication by N such that

$$\gamma^\Omega(N) = -i \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$$

we finally get

$$D\!\!\!\!/^\pm = \begin{pmatrix} \pm\frac{n}{2}\mathrm{Id} & D^\Sigma & 0 & 0 \\ D^\Sigma & \mp\frac{n}{2}\mathrm{Id} & 0 & 0 \\ 0 & 0 & \mp\frac{n}{2}\mathrm{Id} & D^\Sigma \\ 0 & 0 & D^\Sigma & \pm\frac{n}{2}\mathrm{Id} \end{pmatrix}.$$

This expression clearly shows that these operators only depend on the Riemannian metric and the spin structure on Σ. Here the matrix blocks are defined with respect to the decomposition

$$\mathcal{E} \cong \big(\mathbb{S}^+(\Sigma) \oplus \mathbb{S}^-(\Sigma)\big) \oplus \big(\mathbb{S}^+(\Sigma) \oplus \mathbb{S}^-(\Sigma)\big).$$

We summarize the preceding discussion by

Proposition 1. *The bundle $(\mathcal{E}\Omega, \gamma, \nabla)$ is a Dirac bundle equipped with a chirality operator ω whose associated Dirac-type operator $D := \gamma \circ \nabla$ is a first order elliptic differential operator. The restricted triplet $(\mathcal{E}, \slashed{\gamma}, \overline{\nabla})$ is also a Dirac bundle for which the spinorial Gauss formula*

$$\overline{\nabla}_X \Psi = \nabla_X \Psi - \frac{1}{2}\slashed{\gamma}(AX)\Psi \tag{22}$$

holds for all $\Psi \in \Gamma(\mathcal{E})$ and $X \in \Gamma(T\Sigma)$ and such that

$$\slashed{D}\Psi = \frac{n}{2}H\Psi - \gamma(N)D\Psi - \nabla_N\Psi \tag{23}$$

and

$$\slashed{D}\big(\gamma(N)\Psi\big) = -\gamma(N)\slashed{D}\Psi \tag{24}$$

where $\slashed{D} := \slashed{\gamma} \circ \overline{\nabla}$ is the extrinsic Dirac-type operator on $\mathcal{E}$. Moreover, the Dirac-type operators $\slashed{D}^{\pm} := \slashed{D} \pm \frac{n}{2}i\gamma(N)\mathrm{Id}_{\mathcal{E}}$ are first order differential operators which only depend on the Riemannian and spin structures of Σ.

2.3. **The Hyperbolic Reilly formula.** We first recall the hyperbolic version of the Schrödinger-Lichnerowicz formula on the spinor bundle where a proof can be found in [AD], [HMR2] or [M]

$$\int_\Omega \big(\frac{1}{4}\big(R + n(n+1)\big)|\psi|^2 - \frac{n}{n+1}|D_{\pm}^\Omega \psi|^2\big)\, d\Omega \le \int_\Sigma \big(\langle \slashed{D}_{\pm}^\Sigma \psi, \psi \rangle - \frac{n}{2}H|\psi|^2\big)\, d\Sigma$$

for all $\psi \in \Gamma(\mathbb{S}\Omega)$ and where $D_{\pm}^\Omega := D^\Omega \mp \frac{n+1}{2}i\mathrm{Id}$ and $\slashed{D}_{\pm}^\Sigma := \slashed{D}^\Sigma \pm \frac{n}{2}i\gamma^\Omega(N)$. Moreover equality occurs if and only if ψ is a twistor-spinor and the scalar curvature of Ω is constant equal to $-n(n+1)$. Recall that a twistor-spinor on $\mathbb{S}\Omega$ is a smooth spinor field such that $P_X^\Omega \psi = 0$ for all $X \in \Gamma(T\Omega)$ where the operator P^Ω is the twistor operator (also called Penrose operator) defined for all $\psi \in \Gamma(\mathbb{S}\Omega)$ by

$$P_X^\Omega \psi := \nabla_X^\Omega \psi + \frac{1}{n+1}\gamma^\Omega(X)D^\Omega \psi,$$

(for more details, we refer to [BFGK]). We now extend the above Hyperbolic Reilly Inequality to sections of the Dirac bundle $\mathcal{E}\Omega$. For this, we define the twistor operator on $\mathcal{E}\Omega$ by

$$P_X := \nabla_X + \frac{1}{n+1}\gamma(X)D = \begin{cases} P_X^\Omega & \text{if } n+1 = 2m \\[2mm] P_X^\Omega \oplus P_X^\Omega & \text{if } n+1 = 2m+1 \end{cases}$$

and a section $\Psi \in \Gamma(\mathcal{E}\Omega)$ such that $P_X\Psi = 0$ for all $X \in \Gamma(T\Omega)$ will be called a twistor-spinor on $\mathcal{E}\Omega$. Then it is a simple exercise to check that the following formula holds on $\mathcal{E}\Omega$:

Proposition 2. *Let Ω be a compact and connected $(n+1)$-dimensional Riemannian spin manifold with boundary Σ. Assume that the scalar curvature of Ω satisfies $R \ge -n(n+1)$, then for all $\Psi \in \Gamma(\mathcal{E}\Omega)$, we have*

$$-\frac{n}{n+1}\int_\Omega |D^\pm\Psi|^2\, d\Omega \le \int_\Sigma \big(\langle \slashed{D}^\pm\Psi, \Psi \rangle - \frac{n}{2}H|\Psi|^2\big)\, d\Sigma. \tag{25}$$

Moreover equality occurs if and only if Ψ is a twistor-spinor on $\mathcal{E}\Omega$ and $R = -n(n+1)$. Here $\not{D}^\pm$ are defined in (21) and $D^\pm$ are the modified Dirac-type operators acting on sections of $\mathcal{E}\Omega$ defined by:

$$D^\pm := D \mp \frac{n+1}{2} i \operatorname{Id}_{\mathcal{E}\Omega}. \tag{26}$$

2.4. **A boundary-value value problem for the Dirac-type operator D^+.** In this section, we introduce the boundary condition which we will need and prove its ellipticity for a Dirac-type operator acting on $\Gamma(\mathcal{E}\Omega)$. It turns out that this condition is well-known for even dimensional Riemannian spin manifolds: this is the condition associated with a chirality operator (see [HMR1] for example). Here we extend it for odd dimensional Riemannian spin manifolds. Note that, as explained in the previous section, we are not working on the spinor bundle $\mathbb{S}\Omega$ since this boundary condition does not yield to an elliptic boundary condition for the fundamental Dirac operator D^Ω on Ω.

Since the modified Dirac-type operators $D^\pm$ (see (26)) acting on sections of $\mathcal{E}\Omega$ are zero order deformations of the Dirac operator, they define first order elliptic differential operators whose L^2-formal adjoints are $(D^\pm)^* = D^\mp$. This last fact is an obvious consequence of the following integration by parts formula

$$\int_\Omega \langle D\Psi, \Phi \rangle \, d\Omega = \int_\Omega \langle \Psi, D\Phi \rangle \, d\Omega - \int_\Sigma \langle \gamma(N)\Psi, \Phi \rangle \, d\Sigma \tag{27}$$

for all $\Psi, \Phi \in \Gamma(\mathcal{E}\Omega)$ and where $d\Omega$ (resp. $d\Sigma$) is the Riemannian volume element of Ω (resp. Σ). It is then easy to see that we are in the standard setup examined by Bär and Ballmann (see page 5 of [BäBa] for a precise definition of this setting). On the other hand, the fiber preserving endomorphism

$$G = \gamma(N)\omega : \Gamma(\not{\mathcal{E}}) \to \Gamma(\not{\mathcal{E}}),$$

acting on sections of the restricted bundle, is self-adjoint with respect to the pointwise Hermitian scalar product, whose square is the identity. Here ω is the chirality operator defined in Section 2.1. The map G has two eigenvalue ± 1 whose corresponding eigenspaces are interchanged by the isomorphism $\gamma(N)$. Then we consider the two non trivial eigensubbundles $\mathcal{V}^\pm$ over Σ corresponding to the $\pm 1-$eigenvalues of the map G so that the following decomposition holds

$$\not{\mathcal{E}} = \mathcal{V}^+ \oplus \mathcal{V}^-.$$

The pointwise projections on $\mathcal{V}^\pm$ are given by

$$\begin{aligned} P_\pm : L^2(\mathcal{E}\Omega) &\longrightarrow L^2(\mathcal{V}^\pm) \\ \Psi &\longmapsto P_\pm \Psi := \tfrac{1}{2}(\operatorname{Id} \pm \gamma(N)\omega)\Psi, \end{aligned} \tag{28}$$

where $L^2(\mathcal{E}\Omega)$ (resp. $L^2(\mathcal{V}^\pm)$) denotes the space of L^2-integrable sections of $\mathcal{E}\Omega$ (resp. $\mathcal{V}^\pm$). Using the properties (12) and (13) of ω, we easily see that for $X \in \Gamma(T\Sigma)$ and $\Psi \in \Gamma(\not{\mathcal{E}})$, we have $G(\not{\gamma}(X)\Psi) = \mp\not{\gamma}(X)G\Psi$ and then

$\gamma\!\!\!/(X)$ interchanges $\mathcal{V}^+$ and $\mathcal{V}^-$. So from Corollary 7.23 and Proposition 7.24 in [BäBa], we have

Proposition 3. *The pointwise orthogonal decomposition $\mathcal{E} = \mathcal{V}^+ \oplus \mathcal{V}^-$ induces local boundary conditions for D^+. In particular, the operator*

$$D^+ : \mathrm{Dom}(D^+) := \{\Psi \in H_1^2 \,:\, P_\pm(\Psi_{|\Sigma}) = 0\} \longrightarrow L^2(\mathcal{E}\Omega)$$

is Fredholm and if Φ is a smooth section of $\mathcal{E}\Omega$, then any H_1^2-solutions of

$$\begin{cases} D^+\Psi = \Phi & on \, \Omega \\ P_\pm\Psi_{|\Sigma} = 0 & along \, \Sigma, \end{cases}$$

is smooth up to the boundary. Here H_1^2 stands for the Sobolev space of L^2-spinors with weak L^2-covariant derivatives.

It is clear that the same result holds for the Dirac-type operator D^-. Next we only consider the operator D^+ since it is straightforward to check that all the following results also hold for D^-.

Now we want to prove that the Dirac operator D^+ defines an isomorphism between the space

$$\mathrm{Dom}_\pm(D^+) := \{\Psi \in H_1^2(\mathcal{E}\Omega) \,:\, P_\pm\Psi_{|\Sigma} = 0\}$$

onto $L^2(\mathcal{E}\Omega)$, where $P_\pm$ is the projection given by (28). We now denote by $D_\pm^+$ the Dirac-type operator defined on the domain $\mathrm{Dom}_\pm(D^+)$. We have

Proposition 4. *Let Ω be a compact domain with smooth boundary in a $(n+1)$-dimensional Riemannian spin manifold. The Dirac-type operator D^+ with domain $\mathrm{Dom}_\pm(D^+)$ is an isomorphism onto the space of square integrable sections of $\mathcal{E}\Omega$. In particular, for all $\Phi \in \Gamma(\mathcal{E}\Omega)$, there exists a unique smooth section $\Psi \in \Gamma(\mathcal{E}\Omega)$ such that*

$$\begin{cases} D^+\Psi = \Phi & on \, \Omega \\ P_\pm\Psi_{|\Sigma} = 0 & along \, \Sigma. \end{cases} \qquad (29)$$

Proof : From the Stokes' formula (27) we have for all $\Psi, \Phi \in \Gamma(\mathcal{E}\Omega)$

$$\int_\Omega \langle D^+\Psi, \Phi \rangle \, d\Omega = \int_\Omega \langle \Psi, D^-\Phi \rangle \, d\Omega - \int_\Sigma \langle \gamma(N)\Psi, \Phi \rangle \, d\Sigma.$$

On the other hand for all $\Psi \in \Gamma(\mathcal{E}\Omega)$

$$P_\pm\Psi_{|\Sigma} = 0 \iff P_\mp\big(\gamma(N)\Psi_{|\Sigma}\big) = 0,$$

then the boundary term of the previous identity vanishes for all $\Psi \in \mathrm{Dom}_\pm(D^+)$, hence $(D_\pm^+)^* = D_\pm^-$. Since

$$\mathrm{CoKer}(D_\pm^+) \simeq \mathrm{Ker}(D_\pm^+)^* \simeq \mathrm{Ker}(D_\pm^-), \qquad (30)$$

we only have to show that $\mathrm{Ker}(D_\pm^+)$ and $\mathrm{Ker}(D_\pm^-)$ are reduced to zero to conclude that $D_\pm^+$ is an isomorphism. So if $\Psi \in \Gamma(\mathcal{E}\Omega)$ is in the kernel of $D_\pm^+$ that is

$$\begin{cases} D^+\Psi = 0 & on \, \Omega \\ P_\pm\Psi_{|\Sigma} = 0 & along \, \Sigma, \end{cases}$$

then, from the ellipticity of the boundary condition $P_\pm$, it has to be smooth up to the boundary. Moreover since $D^+\Psi = 0$ we have on one hand

$$\int_\Omega \langle D\Psi, \Psi \rangle \, d\Omega = i\frac{n+1}{2} \int_\Omega |\Psi|^2 \, d\Omega,$$

and on the other hand, an integration by parts leads to

$$\int_\Omega \langle D\Psi, \Psi \rangle \, d\Omega = \int_\Omega \langle \Psi, D\Psi \rangle \, d\Omega = -i\frac{n+1}{2} \int_\Omega |\Psi|^2 \, d\Omega.$$

In other words, we showed that

$$(n+1)i \int_\Omega |\Psi|^2 \, d\Omega = 0$$

which implies that $\Psi \equiv 0$ on Ω. We conclude that the kernel of $D_\pm^+$ is trivial and by using a similar argument, we also get that $\mathrm{Ker}(D_\pm^-) = \{0\}$. Then from (30), the operator $D_\pm^+$ is an isomorphism. From this fact, it is obvious to see that for all $\Phi \in \Gamma(\mathcal{E}\Omega)$, there exists a unique smooth solution of (29).

<div align="right">q.e.d.</div>

As a consequence, we prove that the associated non-homogeneous boundary-value problem has a unique smooth solution:

Corollary 5. *Let Σ be a hypersurface bounding a compact domain Ω in an $(n+1)$-dimensional Riemannian spin manifold. Then for all $\Phi \in \Gamma(\mathcal{G})$, there exists a non trivial smooth section $\Psi \in \Gamma(\mathcal{E}\Omega)$, solution of the boundary-value problem*

$$\begin{cases} D^+\Psi = 0 & \text{on } \Omega \\ P_+\Psi_{|\Sigma} = P_+\Phi & \text{along } \Sigma. \end{cases} \quad (31)$$

The same conclusion holds for the boundary condition P_-.

Proof : Let $\widehat{\Phi}$ be a smooth extension of Φ on Ω. From Proposition 4, there exists a smooth solution $\widehat{\Psi} \in \Gamma(\mathcal{E}\Omega)$ to the boundary-value problem

$$\begin{cases} D^+\widehat{\Psi} = -D^+\widehat{\Phi} & \text{on } \Omega \\ P_\pm\widehat{\Psi}_{|\Sigma} = 0 & \text{along } \Sigma. \end{cases}$$

It is then straightforward to see that $\Psi := \widehat{\Psi} + \widehat{\Phi}$ is a smooth section of $\mathcal{E}\Omega$ which satisfies (31). <div align="right">q.e.d.</div>

3. THE HOLOGRAPHIC PRINCIPLE FOR DIRAC BUNDLES

In this section, we prove a holographic principle for the existence of an imaginary Killing spinor. This result is the hyperbolic counterpart of a similar principle for parallel spinor fields proved by the first two authors in [HM1].

Theorem 5. *Let Ω be a compact, connected Riemannian spin manifold with smooth boundary Σ. Assume that the scalar curvature of Ω satisfies $R \geq -n(n+1)k^2$ for some $k > 0$ and the mean curvature H of Σ is positive, then for all $\Phi \in \Gamma(\mathcal{G})$, one has*

$$\int_\Sigma \left(\frac{1}{H} |\mathcal{D}^+\Phi|^2 - \frac{n^2}{4} H |\Phi|^2 \right) d\Sigma \geq 0. \tag{32}$$

Moreover, equality occurs for $\Phi \in \Gamma(\mathcal{G})$ if and only if there exists two imaginary Killing spinor fields Ψ^+, $\Psi^- \in \Gamma(\mathcal{G})$ with Killing number $-(i/2)$ such that $P_+\Psi^+ = P_+\Phi$ and $P_-\Psi^- = P_-\Phi$.

Remark 6. In the previous result, a smooth section $\Phi^\pm \in \Gamma(\mathcal{E}\Omega)$ is called an imaginary Killing spinor on $\mathcal{E}\Omega$ with Killing number $\pm(i/2)$ if it satisfies the equation

$$\nabla_X \Phi^\pm = \pm \frac{i}{2} \gamma(X) \Phi^\pm$$

for all $X \in \Gamma(T\Omega)$. It is clear that if Ω is an even dimensional manifold, the existence of an imaginary Killing spinor on $\mathcal{E}\Omega$ is equivalent to the existence of an imaginary Killing spinor on $\mathbb{S}\Omega$ since in this case, $\mathcal{E}\Omega = \mathbb{S}\Omega$. If the dimension of Ω is odd, the existence of one imaginary Killing spinor with Killing number $\pm(i/2)$ is enough to ensure that $\mathcal{E}\Omega$ carries two imaginary Killing spinors with Killing number $(i/2)$ and $-(i/2)$. Indeed, it is immediate to check that if ϕ denotes such a spinor field on $\mathbb{S}\Omega$, then the fields defined on $\mathcal{E}\Omega$ by $\Phi^+ = (\phi, 0)$ and $\Phi^- = (0, \phi)$ are imaginary Killing spinors on $\mathcal{E}\Omega$ whose Killing number are respectively $\pm(i/2)$ and $\mp(i/2)$. Moreover, they satisfy $|\Phi^\pm|^2 = |\phi|^2$ and they have no zero since imaginary Killing spinors in $\mathbb{S}\Omega$ have no zero (see [Ba1] or [Ba2] for example).

The choice of the boundary condition heavily relies on its behavior with respect to the modified Dirac-type operator $\mathcal{D}^\pm$. We first state the main properties needed here to prove our main result.

Lemma 7. *The Dirac-type operator $\mathcal{D}^\pm$ defined for all $\Phi \in \Gamma(\mathcal{G})$ by:*

$$\mathcal{D}^\pm \Phi := \mathcal{D}\Phi \pm \frac{n}{2} i \gamma(N) \Phi$$

are first order elliptic differential operators which are self-adjoint with respect to the L^2-scalar product on $\mathcal{G}$. Moreover for all $\Phi \in \Gamma(\mathcal{G})$, we have:

$$\mathcal{D}^+(P_\pm \Phi) = P_\mp(\mathcal{D}^+\Phi) \tag{33}$$

and so in particular:

$$\int_\Sigma \langle \mathcal{D}^+\Phi, \Phi \rangle \, d\Sigma = 2 \int_\Sigma \mathrm{Re} \langle \mathcal{D}^+(P_+\Phi), P_-\Phi \rangle \, d\Sigma. \tag{34}$$

Proof : First note that, since $\mathcal{D}^+$ is a zero order deformation of the first order elliptic differential operator $\mathcal{D}$, it is also a first order elliptic operator. Then note that the endomorphism $i\gamma(N)$ of $\mathcal{G}$ is symmetric with respect to

the pointwise Hermitian scalar product $\langle\,,\,\rangle$, so that we easily compute for all $\Phi_1,\ \Phi_2 \in \Gamma(\mathcal{G})$:

$$\int_\Sigma \langle \slashed{D}^+\Phi_1, \Phi_2\rangle\, d\Sigma = \int_\Sigma \langle \slashed{D}\Phi_1 + \frac{n}{2}i\gamma(N)\Phi_1, \Phi_2\rangle\, d\Sigma = \int_\Sigma \langle\Phi_1, \slashed{D}^+\Phi_2\rangle\, d\Sigma$$

since $\slashed{D}$ is L^2-self-adjoint. This proves the first assertion. A straightforward computation shows that $\gamma(N)P_\pm = P_\mp\gamma(N)$ and then the skew-commutativity rule (24) gives (33). Now every section of $\mathcal{G}$ can be decomposed into $\Phi = P_+\Phi + P_-\Phi$ and since this decomposition is pointwise orthogonal, we compute using (33):

$$
\begin{aligned}
\int_\Sigma \langle \slashed{D}^+\Phi, \Phi\rangle\, d\Sigma &= \int_\Sigma \langle P_-(\slashed{D}^+\Phi), P_-\Phi\rangle\, d\Sigma + \int_\Sigma \langle P_+(\slashed{D}^+\Phi), P_+\Phi\rangle\, d\Sigma \\
&= \int_\Sigma \langle \slashed{D}^+(P_+\Phi), P_-\Phi\rangle\, d\Sigma + \int_\Sigma \langle P_-\Phi, \slashed{D}^+(P_+\Phi)\rangle\, d\Sigma \\
&= 2\int_\Sigma \mathrm{Re}\langle \slashed{D}^+(P_+\Phi), P_-\Phi\rangle\, d\Sigma.
\end{aligned}
$$

<div align="right">q.e.d.</div>

Proposition 8. *Let Ω be a compact spin Riemannian manifold with scalar curvature $R \geq -n(n+1)$ and whose boundary Σ has positive mean curvature H. For any section Φ of the restricted Dirac bundle $\mathcal{G}$, one has*

$$0 \leq \int_\Sigma \Big(\frac{1}{H}|\slashed{D}^+P_+\Phi|^2 - \frac{n^2}{4}H|P_+\Phi|^2\Big)\, d\Sigma. \tag{35}$$

Moreover, equality holds if and only if there exists an imaginary Killing spinor $\Psi^+ \in \Gamma(\mathcal{E}\Omega)$ such that $P_+\Psi^+ = P_+\Phi$ along the boundary.

Proof : Take any spinor field $\Phi \in \Gamma(\mathcal{G})$ on the hypersurface and consider the following boundary-value problem

$$\begin{cases} D^+\Psi^+ &= 0 \qquad \text{on } \Omega \\ P_+\Psi^+_{|\Sigma} &= P_+\Phi \qquad \text{on } \Sigma \end{cases} \tag{36}$$

for the Dirac-type operator D^+ and the boundary condition P_+. The existence and uniqueness of a smooth solution $\Psi^+ \in \Gamma(\mathcal{E}\Omega)$ for this boundary-value problem is ensured by Corollary 5. On the other hand, since we assume that $R \geq -n(n+1)$, we can apply the hyperbolic Reilly inequality (25) to Ψ^+ to get the following inequality

$$0 \leq \int_\Sigma \Big(\langle \slashed{D}^+\Psi^+, \Psi^+\rangle - \frac{n}{2}H|\Psi^+|^2\Big)\, d\Sigma.$$

This inequality combined with (34), imply

$$0 \leq \int_\Sigma \Big(2\mathrm{Re}\langle \slashed{D}^+P_+\Psi^+, P_-\Psi^+\rangle - \frac{n}{2}H|P_+\Psi^+|^2 - \frac{n}{2}H|P_-\Psi^+|^2\Big)\, d\Sigma. \tag{37}$$

Since we assume that the mean curvature $H > 0$, we can write

$$0 \leq |\sqrt{\frac{2}{nH}}\,\rlap{D}{/}\,^+ P_+ \Psi^+ - \sqrt{\frac{nH}{2}}\,P_- \Psi^+|^2 =$$

$$\frac{2}{nH}|\rlap{D}{/}\,^+ P_+ \Psi^+|^2 + \frac{nH}{2}|P_- \Psi^+|^2 - 2\mathrm{Re}\langle \rlap{D}{/} P_+ \Psi^+, P_- \Psi^+ \rangle.$$

In other words, we have

$$2\mathrm{Re}\langle \rlap{D}{/}\,^+ P_+ \Psi^+, P_- \Psi^+ \rangle - \frac{nH}{2}|P_- \Psi^+|^2 \leq \frac{2}{nH}|\rlap{D}{/}\,^+ P_+ \Psi^+|^2,$$

which, when combined with Inequality (37), knowing that $P_+ \Psi^+ = P_+ \Phi$, imply Inequality (35).

Assume now that equality is achieved, then the spinor field $\Psi^+ \in \Gamma(\mathcal{E}\Omega)$ which satisfies the boundary-value problem (36) is in fact a twistor-spinor since we have equality in the hyperbolic Reilly formula (25). Moreover, since the condition $D^+ \Psi^+ = 0$ translates to $D\Psi^+ = \frac{n+1}{2}i\Psi^+$, the section Ψ^+ is in fact an imaginary Killing spinor on $\mathcal{E}\Omega$ with Killing number $-(i/2)$. Moreover, it is obvious that $P_+ \Psi^+_{|\Sigma} = P_+ \Phi$ as asserted.

Conversely, if Ψ^+ is an imaginary Killing spinor on $\mathcal{E}\Omega$ then from (23) we compute

$$\begin{aligned} \rlap{D}{/}\Psi^+ &= \frac{n}{2}H\Psi^+ - \gamma(N)D\Psi^+ - \nabla_N \Psi^+ \\ &= \frac{n}{2}H\Psi^+ - \frac{n}{2}i\gamma(N)\Psi^+ \end{aligned}$$

which can be written as $\rlap{D}{/}\,^+ \Psi^+ = \frac{n}{2}H\Psi^+$. Now we decompose the section Ψ^+ with respect to P_+ and P_- and thus the relation (33) yields

$$\rlap{D}{/}\,^+ (P_\pm \Psi^+) = \frac{n}{2}H P_\mp \Psi^+. \tag{38}$$

Moreover, from the L^2-self-adjointness of $\rlap{D}{/}\,^+$ and (38), we get

$$\begin{aligned} \frac{n}{2}\int_\Sigma H|P_- \Psi^+|^2\,d\Sigma &= \int_\Sigma \langle \rlap{D}{/}\,^+ P_+ \Psi^+, P_- \Psi^+ \rangle\,d\Sigma \\ &= \int_\Sigma \langle P_+ \Psi^+, \rlap{D}{/}\,^+ P_- \Psi^+ \rangle\,d\Sigma \\ &= \frac{n}{2}\int_\Sigma H|P_+ \Psi^+|^2\,d\Sigma \end{aligned}$$

that is

$$\int_\Sigma H|P_- \Psi^+|^2\,d\Sigma = \int_\Sigma H|P_+ \Psi^+|^2\,d\Sigma. \tag{39}$$

Finally, using (38) and (39), it follows

$$\int_\Sigma \left(\frac{1}{H}|\rlap{D}{/} P_+ \Psi^+|^2 - \frac{n^2}{4}H|P_+ \Psi^+|^2\right)d\Sigma = \frac{n^2}{4}\int_\Sigma H(|P_- \Psi^+|^2 - |P_+ \Psi^+|^2)\,d\Sigma$$

so that equality is achieved in (35).

q.e.d.

We can mimic this proof step by step to get the counterpart of this result for the boundary condition P_-

Proposition 9. *Let Ω be a compact spin Riemannian manifold with scalar curvature $R \geq -n(n+1)$, whose boundary Σ has positive mean curvature H. For any section Φ of the restricted Dirac bundle $\mathcal{E}$, one has*

$$0 \leq \int_{\Sigma} \Big(\frac{1}{H}|\not{D}^+ P_- \Phi|^2 - \frac{n^2}{4}H|P_-\Phi|^2\Big)\, d\Sigma. \tag{40}$$

Moreover, equality holds if and only if, there exists an imaginary Killing spinor Ψ^- on $\mathcal{E}\Omega$ such that $P_-\Psi^- = P_-\Phi$, along the boundary.

Proof of **Theorem 5***:* By Propositions 8 and 9, the field $\Phi \in \Gamma(\mathcal{E})$ satisfies inequalities (35) and (40). Summing these estimates and using the relation (33) gives the result. The equality case also follows directly from the characterization of the equality cases in Propositions 8 and 9.

<div align="right">q.e.d.</div>

Now making use of the restriction to the hypersurface of an imaginary Killing spinor field, we get

Theorem 6. *Let (Ω^{n+1}, g) be a compact, connected $(n+1)$-dimensional Riemannian spin manifold with smooth boundary Σ. Assume that the scalar curvature of Ω satisfies $R \geq -n(n+1)$ and that the mean curvature H of Σ is positive. Suppose furthermore that Σ admits an isometric and isospin immersion F into another $(n+1)$-dimensional Riemannian spin manifold (Ω_0, g_0) endowed with a non trivial $\pm(i/2)$-imaginary Killing spinor field $\Phi^\pm \in \Gamma(\mathcal{E}\Omega_0)$ and denote by H_0 the mean curvature of this immersion. Then the following inequality holds*

$$\int_{\Sigma} \Big(\frac{H_0^2 - H^2}{H}\Big)|\Phi^\pm|^2\, d\Sigma \geq 0 \tag{41}$$

and equality occurs if and only if both immersions have the same shape operators and Σ is connected.

Proof of **Theorem 6** *:* We only consider the case where $\Phi^- \in \Gamma(\mathcal{E}\Omega_0)$ is an imaginary Killing spinor with Killing number $-(i/2)$. If Σ_0 is a connected component of the boundary Σ, then, by taking the restriction of the imaginary Killing spinor $\Phi^- \in \Gamma(\mathcal{E}\Omega_0)$ to Σ_0, we get the existence of a section $\Phi_0^- := \Phi_{|\Sigma_0}^-$ which satisfies the intrinsic Dirac-type equation

$$\not{D}^+ \Phi_0^- = \frac{n}{2}H_0\Phi_0^-. \tag{42}$$

Now we extend the section Φ_0^- on Σ in such a way that its extension, also denoted by $\Phi_0^- \in \Gamma(\mathcal{E})$, vanishes on $\Sigma - \Sigma_0$. Then putting this spinor field into (32) gives the estimate (41). Assume now that equality is achieved, then from the equality case of (32), there exists two imaginary Killing spinor fields

Ψ^+, $\Psi^- \in \Gamma(\mathcal{E}\Omega)$ with Killing number $-(i/2)$ such that $P_+\Psi^+ = P_+\Phi_0^-$ and $P_-\Psi^- = P_-\Phi_0^-$. Using (42), (33) and Formula (23), we have

$$H_0 P_+\Phi_0^- = \frac{2}{n}\slashed{D}^+(P_-\Phi_0^-) = \frac{2}{n}\slashed{D}^+(P_-\Psi^-) = H P_+\Psi^-. \qquad (43)$$

Similarly, we obtain

$$H_0 P_-\Phi_0^- = \frac{2}{n}\slashed{D}^+(P_+\Phi_0^-) = \frac{2}{n}\slashed{D}^+(P_+\Psi^+) = H P_-\Psi^+. \qquad (44)$$

Applying the operator $\slashed{D}$ to the first and last terms of (43), we get

$$\slashed{\gamma}(\nabla^\Sigma H_0)P_+\Phi_0^- + \frac{n}{2}H_0^2 P_-\Phi_0^- = \slashed{\gamma}(\nabla^\Sigma H)P_+\Psi^- + \frac{n}{2}H^2 P_-\Psi^-$$

which, using again the equalities above, finally gives

$$\slashed{\gamma}(\nabla^\Sigma H_0)P_+\Phi_0^- + \frac{n}{2}H_0^2 P_-\Phi_0^- = \frac{H_0}{H}\slashed{\gamma}(\nabla^\Sigma H)P_+\Phi_0^- + \frac{n}{2}H^2 P_-\Phi_0^-.$$

The same argument applied to (44) yields

$$\slashed{\gamma}(\nabla^\Sigma H_0)P_-\Phi_0^- + \frac{n}{2}H_0^2 P_+\Phi_0^- = \frac{H_0}{H}\slashed{\gamma}(\nabla^\Sigma H)P_-\Phi_0^- + \frac{n}{2}H^2 P_+\Phi_0^-,$$

so that the sum of the last two formulae implies

$$\slashed{\gamma}(\nabla^\Sigma H_0)\Phi_0^- + \frac{n}{2}H_0^2 \Phi_0^- = \frac{H_0}{H}\slashed{\gamma}(\nabla^\Sigma H)\Phi_0^- + \frac{n}{2}H^2 \Phi_0^-.$$

Moreover, since the spinor fields $\slashed{\gamma}(\nabla^\Sigma H_0)\Phi_0^-$ and $\slashed{\gamma}(\nabla^\Sigma H)\Phi_0^-$ are both orthogonal to Φ_0^-, and since the spinor Φ_0^- has no zeros on Σ_0 (see Remark 6), we deduce that $H_0^2 = H^2$ and $H\nabla^\Sigma H_0 = H_0\nabla^\Sigma H$. From these facts, we conclude that H_0 has no zeros since H is positive and so we may assume that $H_0 = H$. Using this equality in (43) and (44) gives $\Phi_{0|\Sigma}^- = \Psi_{|\Sigma}^+ = \Psi_{|\Sigma}^-$. By definition, we have $\Phi_{0|\Sigma-\Sigma_0} = 0$ on $\Sigma - \Sigma_0$, thus

$$P_+\Psi^+ = P_+\Phi_0^- = 0 \quad \text{and} \quad P_-\Psi^- = P_-\Phi_0^- = 0.$$

Applying the operator $\slashed{D}^+$ to these equalities and using (33) and (23), we get

$$0 = \slashed{D}^+(P_+\Psi^+) = \frac{n}{2}H P_-\Psi^+, \quad 0 = \slashed{D}^+(P_-\Psi^-) = \frac{n}{2}H P_+\Psi^-$$

and since $H > 0$, we deduce

$$\Psi_{|\Sigma-\Sigma_0}^+ = \Psi_{|\Sigma-\Sigma_0}^- = 0.$$

However, since Ψ^+ and Ψ^- are imaginary Killing spinors on $\mathcal{E}\Omega$, they have no zeros, so this is impossible unless $\Sigma = \Sigma_0$ is connected.

Finally, as another consequence of the preceding argument, we have that Φ_0^- is the restriction to Σ of Ψ^+ (and Ψ^-) via the embedding of Σ as the

boundary of Ω and of Φ^- via the immersion of Σ in Ω_0. Then we can apply the spinorial Gauss formula (22) for the first immersion, that is

$$\slashed{\nabla}_X \Psi^+ = -\frac{i}{2}\gamma(X)\Psi^+ - \frac{1}{2}\slashed{\gamma}(AX)\Psi^+ \qquad (45)$$

for all $X \in \Gamma(T\Sigma)$ (here A is the second fundamental form of $\Sigma \hookrightarrow \Omega$), as well as

$$\slashed{\nabla}_X \Phi_0^- = -\frac{i}{2}\gamma^0(X)\Phi_0^- - \frac{1}{2}\slashed{\gamma}(A_0 X)\Phi_0^- \qquad (46)$$

for the second immersion $\Sigma \hookrightarrow \Omega_0$. The notation γ^0 stands for the Clifford multiplication on $\mathcal{E}\Omega_0$. Now we claim that

$$\gamma(X)\Psi^+ = \gamma^0(X)\Phi_0^- \qquad (47)$$

for all $X \in \Gamma(T\Sigma)$. Indeed from Section 2.2, we have seen that we can choose $\gamma^0(N_0)$ and $\gamma(N)$ such that $\gamma^0(N_0)\Phi = \gamma(N)\Phi$ for all $\Phi \in \Gamma(\mathcal{G})$ and thus for all $X \in \Gamma(T\Sigma)$,

$$\gamma(X)\Phi = -\slashed{\gamma}(X)\gamma(N)\Phi = -\slashed{\gamma}(X)\gamma^0(N_0)\Phi = \gamma^0(X)\Phi.$$

Using the fact that $\Psi_{|\Sigma}^+ = \Phi_{0|\Sigma}^-$ in (45) and (46) with the relation (47) finally give

$$\slashed{\gamma}(A_0 X - AX)\Phi_{0|\Sigma}^- = 0$$

for X tangent to Σ, and since Φ_0^- has no zeros, we get $A_0 = A$.

The converse is clear. If the two shape operators A and A_0 coincide, then the corresponding traces nH and nH_0 taken with respect to the common induced metric should be equal. Then we have equality in (41).

<div align="right">q.e.d.</div>

4. A NEW QUASI-LOCAL MASS

We propose here a local version of the positive mass theorem obtained by Wang [Wa1] and Chruściel-Herzlich [CH] for asymptotically hyperbolic manifolds.

4.1. **The Hyperbolic space and Hypersurfaces.** In this section, we recall some well known facts regarding imaginary Killing spinors of the hyperbolic space $\mathbb{H}^{n+1}$. The classification of complete manifolds carrying an imaginary Killing spinor has been obtained by H. Baum in [Ba1, Ba2] (see Remark 10 below). A standard model of the hyperbolic space is the unit ball $\mathbb{B}^{n+1}$ endowed with the Riemannian metric $g_{\mathbb{H}} = f^2 g_{\mathbb{E}}$ where $g_{\mathbb{E}}$ is the Euclidean metric and $f(x) = 2/(1 - |x|_{\mathbb{E}}^2)$ for $x \in \mathbb{B}^{n+1}$. Here $|\,.\,|_{\mathbb{E}}$ denotes the Euclidean norm associated to $g_{\mathbb{E}}$. Since the Riemannian metrics $g_{\mathbb{H}}$ and $g_{\mathbb{E}}$ are conformally related, we can canonically identify the corresponding

spinor bundles $\mathbb{SH}$ and $\mathbb{SB}$. Now we consider the $\mathbb{C}^N$-valued constant function on $\mathbb{B}^{n+1}$ equal to $a \in \mathbb{C}^N$, with $N = 2^{[\frac{n+1}{2}]}$, which allows to define a spinor field on the unit Euclidean ball, by setting

$$\psi_a^{\pm}(x) := f^{\frac{1}{2}}(x)(\mathrm{Id} \pm i\gamma^{\mathbb{E}}(x))a. \tag{48}$$

The spinor field $\psi_a^{\pm}$ induces on $\mathbb{H}^{n+1}$ an imaginary Killing spinor field also denoted by $\psi_a^{\pm}$. In fact, every imaginary Killing spinor on the hyperbolic space can be obtained in such a way.

Remark 10. It is a well-known fact that, after suitable rescaling of the metric, an $(n+1)$-dimensional manifold P with an imaginary Killing spinor has to be Einstein with Ricci curvature $-n$. If P is complete, H. Baum proved in [Ba1, Ba2] that it has to be a warped product $\mathbb{R} \times_{\exp} P_0$, i.e. the manifold $\mathbb{R} \times P_0$ is endowed with the metric

$$g := dt^2 \oplus e^{2t} g_{P_0}$$

where (P_0, g_{P_0}) is an n-dimensional complete Riemannian spin manifold admitting a non-trivial parallel spinor. In case P_0 is the Euclidean space $\mathbb{R}^n$, then P is nothing but the hyperbolic space with constant curvature -1.

In the following, Σ is a smooth oriented hypersurface in $\mathbb{H}^{n+1}$ whose Weingarten map is denoted by A_0, i.e., $A_0(X) = -\nabla_X^{\mathbb{H}} N_0$ for all $X \in \Gamma(T\Sigma)$, here $\nabla^{\mathbb{H}}$ is the Levi-Civita connection on $\mathbb{H}^{n+1}$ and N_0 is the associated unit inward normal. Now we discuss the existence of *imaginary Killing spinors* on $\mathcal{E}\mathbb{H}$ as defined in Remark 6 and its consequences. For $n+1$ even, the bundle $\mathcal{E}\mathbb{H}$ corresponds to the spinor bundle over the hyperbolic space and this situation is well-known. For the sake of completeness, we include a brief discussion of this case.

The even dimensional case

In this case, the bundle $\mathcal{E}\mathbb{H}$ is simply the standard spinor bundle on $\mathbb{H}^{2m}$ with Clifford multiplication $\gamma^0 = \gamma^{\mathbb{H}}$ and Levi-Civita connection $\nabla^0 = \nabla^{\mathbb{H}}$. Then by the spin Gauss formula (22), the restriction of an imaginary Killing spinor $\psi_a^{\pm} \in \Gamma(\mathcal{E}\mathbb{H})$ to Σ satisfies:

$$\overline{\nabla}_X \psi_a^{\pm} = \pm \frac{i}{2} \gamma^0(X)\psi_a^{\pm} - \frac{1}{2}\gamma(A_0 X)\psi_a^{\pm}.$$

In particular, $\psi_a^{\pm}$ is a solution of the Dirac-type equation

$$\slashed{D}\psi_a^{\pm} = \pm \frac{i}{2}\gamma^0(N_0)\psi_a^{\pm} + \frac{n}{2}H_0\psi_a^{\pm}$$

which, by the discussion in Section 2.2, translates in an intrinsic way to Σ by

$$\slashed{D}^{\mp}\psi_a^{\pm} = \frac{n}{2}H_0\psi_a^{\pm}.$$

The odd dimensional case

If we assume now that $n = 2m$, the vector bundle $\mathcal{E}\mathbb{H}$ is simply two copies of the spinor bundle $\mathbb{SH}$ with Clifford multiplication $\gamma^0 = \gamma^{\mathbb{H}} \oplus -\gamma^{\mathbb{H}}$ and

spin Levi-Civita connections $\nabla^0 = \nabla^{\mathbb{H}} \oplus \nabla^{\mathbb{H}}$. In this situation, from the discussion in Remark 6, the spinor field defined by $\Psi_a^\pm := (\psi_a^\pm, 0) \in \Gamma(\mathcal{E}\mathbb{H})$, where $\psi_a^\pm \in \Gamma(\mathbb{S}\mathbb{H})$ is given by (48), satisfies

$$\nabla_X \Psi_a^\pm = \pm \frac{i}{2} \gamma(X) \Psi_a^\pm$$

for all $X \in \Gamma(T\mathbb{H})$ so that it is an imaginary Killing spinor field on $\mathcal{E}\mathbb{H}$ which, in addition, satisfies $|\Psi_a^\pm|^2 = |\psi_a^\pm|^2$. In fact, the spinor field $\Psi_a^\pm$ is characterized by

$$\Psi_a^\pm = f^{\frac{1}{2}}(x)(\mathrm{Id} \pm i\gamma_e(x))a$$

where $\gamma_e := \gamma^{\mathbb{E}} \oplus -\gamma^{\mathbb{E}}$ is the Clifford multiplication on the trivial Dirac bundle $\mathcal{E}\mathbb{B}$ and $a \in \mathbb{C}^N$ is identified with $(a, 0) \in \mathbb{C}^N \oplus \mathbb{C}^N$. Now recall from Section 2.2 that the restricted bundle $\mathcal{\cancel{E}} := \mathcal{E}\mathbb{H}_{|\Sigma}$ can be identified with $\mathbb{S}\Sigma \oplus \mathbb{S}\Sigma$, Clifford multiplication $\cancel{\gamma} = \gamma^\Sigma \oplus \gamma^\Sigma$ and spin Levi-Civita connection $\cancel{\nabla} = \nabla^\Sigma \oplus \nabla^\Sigma$. From these identifications, it is straightforward, using the spinorial Gauss formula (22) and the definition of the Dirac-type operator $\cancel{D}^\pm$, to check that

$$\cancel{D}^\mp \Psi_a^\pm = \frac{n}{2} H_0 \Psi_a^\pm$$

which, by Section 2.2, only depend on the Riemannian metric and the spin structure on Σ.

The previous results could be stated as :

Proposition 11. *For any $a \in \mathbb{C}^N$, the sections of $\mathcal{E}\mathbb{H}$ defined by*

$$\Phi_a^\pm := \begin{cases} \psi_a^\pm & \text{if } n \text{ is odd} \\[2mm] \Psi_a^\pm & \text{if } n \text{ is even} \end{cases}$$

are imaginary Killing spinors on $\mathcal{E}\mathbb{H}$. Moreover, if Σ is an oriented hypersurface in $\mathbb{H}^{n+1}$, then $\Phi_a^\pm$ satisfies

$$\cancel{D}^\mp \Phi_a^\pm = \frac{n}{2} H_0 \Phi_a^\pm$$

and this equation only depends on the Riemannian and spin structures of Σ.

As we will see in the next section, the proof of Theorem 3 relies essentially on (32). However, as easily seen, this principle depends strongly on spinor data whereas our energy-momentum vector $\mathbf{E}(\Sigma)$ does not. A trick by Wang (p. 285-286 in [Wa1]), generalized by Kwong (Proposition 2.1 and 2.2 in [K]), allows to clarify these aspects. Indeed, since for any imaginary Killing $\Phi_a^\pm \in \Gamma(\mathcal{E}\mathbb{H})$ as in Proposition 11, we have $|\Phi_a^\pm|^2 = |\psi_a^\pm|^2$, we easily deduce

Lemma 12. *For every imaginary Killing spinor $\Phi_a^\pm \in \Gamma(\mathcal{E}\mathbb{H})$, there exists a vector field $\zeta_a^\pm \in \mathbb{R}^{n+1,1}$ given by*

$$\zeta_a^\pm = \mp i \sum_{j=1}^{n+1} \langle \gamma_e(\partial_{x_j})a, a \rangle \partial_{x_j} - |a|^2 \partial_t$$

such that

$$|\Phi_a^\pm|^2 = -2\langle \mathbf{X}, \zeta_a^\pm \rangle_{\mathbb{R}^{n+1,1}}. \tag{49}$$

Moreover, for every null vector $\zeta = (\zeta_1, \cdots, \zeta_{n+1}, 1) \in \mathbb{R}^{n+1,1}$, there exists $a \in \mathbb{C}^N$ with $|a| = 1$ such that $\zeta = \zeta_a^\pm$. Here $\mathbf{X} = (x_1, ..., x_{n+1}, t)$ is the position vector field in the Minkowski spacetime and γ_e is Clifford multiplication on the Dirac bundle $\mathcal{E}\mathbb{B}$.

4.2. **Non-negativity of the quasi-local mass.** In this section, we prove Theorem 3. More precisely, we have to show that the energy-momentum vector field $\mathbf{E}(\Sigma) \in \mathbb{R}^{n+1,1}$ defined by (2) is timelike future directed or zero. For this we first recall a characterization of such vector fields given in Lemma 5.2 of [WY1] for 3-dimensional manifolds but which is easily seen to be true in any dimension.

Lemma 13. *A non-zero vector $v = (v_1, \cdots, v_{n+1}, w)$ is timelike future directed if and only if $\langle v, \zeta \rangle < 0$ for all $\zeta = (\zeta_1, \cdots, \zeta_{n+1}, 1)$ with $\sum_{j=1}^{n+1} \zeta_j^2 = 1$.*

From this characterization, we first have to prove that $\langle \mathbf{E}(\Sigma), \zeta \rangle < 0$ for all null vectors $\zeta = (\zeta_1, \cdots, \zeta_{n+1}, 1)$, that is

$$\int_\Sigma \left(\frac{H_0^2 - H^2}{H} \right) \langle X, \zeta \rangle \, d\Sigma < 0 \tag{50}$$

unless $\mathbf{E}(\Sigma) = 0$. However, Lemma 12 ensures that for any null vector ζ as above there exits $a \in \mathbb{C}^N$ with $|a| = 1$ such that $\zeta = \zeta_a^\pm$. Then from (49), the inequality (50) is equivalent to

$$\int_\Sigma \left(\frac{H_0^2 - H^2}{H} \right) |\Phi_a^\pm|^2 \, d\Sigma > 0 \tag{51}$$

for all $a \in \mathbb{C}^N$ with $|a| = 1$. On the other hand, since we assume that Σ admits an isometric and isospin immersion F into the hyperbolic space $\mathbb{H}^{n+1}$, by Proposition 11 it follows that every imaginary Killing spinor field of the form $\Phi_a^\pm$ induces a solution of the Dirac-type equation $\slashed{D}^\mp \Phi_a^\pm = \frac{n}{2} H_0 \Phi_a^\pm$ on Σ (intrinsically to Σ). Moreover since we assume that H is positive on Σ, we can apply (32) to every $\Phi_a^\pm$, to get (51).

However, if equality is achieved, it follows from Theorem 6 that the shape operators of Σ with respect to its embedding in Ω and its immersion in $\mathbb{H}^{n+1}_{-k^2}$ are the same so that $\mathbf{E}(\Sigma) = 0$. This implies that $\mathbf{E}(\Sigma)$ is timelike future directed or zero.

Suppose now that $\mathbf{E}(\Sigma) = 0$. In this case, we already know that Σ is connected and that the second fundamental form A of Σ in Ω agrees with the one of Σ in $\mathbb{H}^{n+1}$ denoted by A_0. On the other hand, the hyperbolic space $\mathbb{H}^{n+1}$ admits a maximal number of linearly independent imaginary Killing spinor fields, so that we can repeat the argument in the proof of Theorem 6 for each one of the restrictions to Σ of these spinor fields. In this way we obtain a maximal number of imaginary Killing spinor fields defined on Ω. But, according to [Ba2] (see also [BFGK]), this forces the manifold

Ω to have constant curvature -1. Moreover, since $A = A_0$, we can then glue along Σ in Ω the exterior of Σ in the hyperbolic space to obtain a smooth complete Riemannian manifold M with constant negative sectional curvature which is isometric to the hyperbolic space at infinity. We easily conclude that M is isometric to $\mathbb{H}^{n+1}$ and then Ω is isometric to a compact domain of $\mathbb{H}^{n+1}$. Finally we may apply the fundamental theorem of the local theory of hypersurfaces (see Theorem 2.1 in [AKY]) to deduce that the embedding of Σ in Ω and its immersion in $\mathbb{H}^{n+1}$ are congruent. The converse of the equality case in Theorem 3 is straightforward.

4.3. **The two dimensional case.** In this section, we consider the case $n = 2$ which is the most relevant from a physical point of view. More precisely, we propose to define a new notion of local energy-momentum vector by setting

$$\mathbf{E}(\Sigma) = \int_{\Sigma} \frac{H_0^2 - H^2}{H} \mathbf{X} \, d\Sigma,$$

with (Σ, g) a topological 2-sphere, whose Gauss curvature $K > -k^2$ and mean curvature $H > 0$, considered as the boundary of a 3-dimensional compact Riemannian domain Ω with scalar curvature $R \geq -6k^2$. Here H_0 is the mean curvature of the embedding of (Σ, g) into the standard hyperbolic space $\mathbb{H}^3_{-k^2}$ (whose existence and uniqueness are proved in [P] and [DCW]). Then it follows easily from Theorem 3, that if Ω is not isometric to a domain of $\mathbb{H}^3_{-k^2}$, then the energy-momentum vector $\mathbf{E}(\Sigma)$ is a timelike future directed vector in $\mathbb{R}^{3,1}$. Moreover, it is zero if and only if Ω is a domain in the hyperbolic space. For a more precise statement of this result, we refer to Theorem 4.

We conclude that $\mathbf{E}(\Sigma)$ has the non negativity and rigidity properties which are needed to define an appropriate notion of quasilocal mass. Another important feature is also required: the limit of $\mathbf{E}(\Sigma)$ should recover the total energy in the asymptotically hyperbolic case. So let us first recall this setting as well as a notion of total energy for such manifolds defined by Wang [Wa1]. A more general setting is described in [CH]. A complete non compact Riemannian manifold (M^3, g) is asymptotically hyperbolic (AH) if M is the interior of a compact manifold $\overline{M}$ with boundary $\partial \overline{M}$ such that

(1) there is a smooth function r on $\overline{M}$, with $r > 0$ on M and $r = 0$ on $\partial \overline{M}$, such that $\overline{g} = r^2 g$ extends as a smooth Riemannian metric on $\overline{M}$;

(2) $|dr|_{\overline{g}} = 1$ on $\partial \overline{M}$;

(3) $\partial \overline{M}$ is the standard unit sphere $\mathbb{S}^2$;

(4) on a collar neighborhood of $\partial \overline{M}$, we have:

$$g = \sinh^{-2}(r)(dr^2 + g_r),$$

where g_r is an r-dependent family of metrics on $\mathbb{S}^2$ such that:

$$g_r = g_0 + \frac{r^3}{3}h + e.$$

Here g_0 is the standard metric on the sphere, h is a smooth symmetric 2-tensor on $\mathbb{S}^2$ and e is of order $O(r^4)$.

The following positive mass theorem was proved by Wang:

Theorem 7. [Wa1] *If (M^3, g) is an (AH) Riemannian manifold such that its scalar curvature satisfies $R \geq -6$, then the energy-momentum vector*

$$\Upsilon = \left(\int_{\mathbb{S}^2} \mathrm{tr}_{g_0}(h)d\mathbb{S}, \int_{\mathbb{S}^2} \mathrm{tr}_{g_0}(h)x d\mathbb{S} \right) \in \mathbb{R}^{3,1}$$

is timelike future directed or zero. It is zero if and only if (M^3, g) is isometric to the hyperbolic space $\mathbb{H}^3$. Here $d\mathbb{S}$ denotes the standard Riemannian measure on the round sphere.

Using a recent work of Kwong and Tam [KT], we show that our local energy-momentum vector (under some additional technical assumptions) converges to the energy-momentum vector Υ. In fact, as in [KT], we assume that the following hold:

(A) $\qquad\qquad \nabla_{\mathbb{S}^2}e, \nabla_{\mathbb{S}^2}^2 e, \nabla_{\mathbb{S}^2}^3 e, \nabla_{\mathbb{S}^2}^4 e$ and $\frac{\partial e}{\partial r}$ are of order $O(r^3)$

where $\nabla_{\mathbb{S}^2}^k$ is the Levi-Civita connection of order k on tensor fields. Then consider a geodesic sphere $S_r \subset (M, g)$ for r small and let H be its mean curvature. We identify S_r as the standard sphere $\mathbb{S}^2$ with metric γ_r induced from g. For r small enough, the Gauss curvature of (S_r, γ_r) is positive, hence (S_r, γ_r) can be isometrically embedded into $\mathbb{H}^3$ by Pogorelov's Theorem. If $X^{(r)}$ denotes this embedding and if o_r is the center of the largest geodesic sphere contained in the interior of $X^{(r)}(S_r)$, then Kwong and Tam prove that we can choose the center of the geodesic balls at a fixed point $o \in \mathbb{H}^3$. In addition to this, they construct isometries ι_r of $\mathbb{H}^3$ fixing o such that, when $X^{(r)}$ is seen as an embedding of (S_r, γ_r) into $\mathbb{R}^{3,1}$ (via $\mathbb{H}^3$), the following expansions hold

$$
\left\{
\begin{aligned}
H &= \cosh r - \tfrac{1}{4}r^3 \mathrm{tr}_{g_0}(h) + o(r^4) = 1 + \tfrac{r^2}{2} - \tfrac{1}{4}r^3 \mathrm{tr}_{g_0}(h) + o(r^4) \\[4pt]
H_0 &= \cosh r + o(r^4) = 1 + \tfrac{r^2}{2} + o(r^3) \\[4pt]
dS_r &= \left(\tfrac{1}{\sinh^2 r} + o(r^2) \right) d\mathbb{S} = \left(\tfrac{1}{r^2} + o(\tfrac{1}{r}) \right) d\mathbb{S} \\[4pt]
\iota_r \circ X^{(r)}(x) &= \left(\tfrac{1}{r} + o(1), \tfrac{x}{r} + o(\tfrac{1}{r}) \right)
\end{aligned}
\right.
$$

Form these estimates, straightforward calculations show that

$$\frac{H_0^2 - H^2}{H} = \frac{1}{2}\mathrm{tr}_{g_0}(h)r^3 + o(r^3)$$

and

$$\iota_r \circ X^{(r)}(x) \, dS_r = \Big(\frac{1}{r^3} + o(\frac{1}{r^2}), \frac{x}{r^3} + o(\frac{1}{r^3}) \Big) \, d\mathbb{S},$$

hence,

Theorem 8. *Let* (M^3, g) *be a* 3-*dimensional (AH) hyperbolic manifold satisfying the assumptions* **(A)**, *then:*

$$\lim_{r \to 0} \mathbf{E}(S_r) = \frac{1}{2} \Upsilon$$

where

$$\mathbf{E}(S_r) = \int_{S_r} \Big(\frac{H_0^2 - H^2}{H} \Big) \iota_r \circ X^{(r)} dS_r.$$

As we have seen, the proof of Theorem 4 makes no use of the Postive Mass Theorem for (AH) manifolds unlike the results of Shi-Tam and Kwong. In fact, combining Theorems 4 and 8, we get an alternative proof of the Positive Mass Theorem of Wang under the additional assumptions **(A)**.

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